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Horizon shells and BMS-like soldering transformations

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Abstract: We revisit the theory of null shells in general relativity, with a particular emphasis on null shells placed at horizons of black holes. We study in detail the considerable freedom that is available in the case that one solders two metrics together across null hypersurfaces (such as Killing horizons) for which the induced metric is invariant under translations along the null generators. In this case the group of soldering transformations turns out to be infinite dimensional, and these solderings create non-trivial horizon shells containing both massless matter and impulsive gravitational wave components. We also rephrase this result in the language of Carrollian symmetry groups. To illustrate this phenomenon we discuss in detail the example of shells on the horizon of the Schwarzschild black hole (with equal interior and exterior mass), uncovering a rich classical structure at the horizon and deriving an explicit expression for the general horizon shell energy-momentum tensor. In the special case of BMS-like soldering supertranslations we find a conserved shell-energy that is strikingly similar to the standard expression for asymptotic BMS supertranslation charges, suggesting a direct relation between the physical properties of these horizon shells and the recently proposed BMS supertranslation hair of a black hole.

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1 Introduction

In this paper we revisit the theory of null shells in general relativity [1]–[4], with a particular emphasis on null shells placed at horizons of black holes.

It seems to be a somewhat underappreciated fact that (in contrast to what happens for spacelike or timelike shells) there can be considerable freedom in "soldering" two geometries along a given null hypersurface while maintaining the Israel junction condition that the induced metric is continuous across the surface. Even though this was pointed out in [1], and isolated examples were already known in the literature before (e.g. the Dray 't Hooft shell separating (or joining) two equal mass Schwarzschild black holes along their horizon [5]), there seems to have been no systematic subsequent analysis of this phenomenon.

We therefore begin with a systematic analysis of the conditions under which non-trivial soldering transformations can exist, and we also place these results into the general setting of Carrollian manifolds and associated notions of symmetry groups, as recently formulated in [6, 7].

It follows from this analysis that the resulting soldering group is infinite-dimensional when the induced metric on the null hypersurface is invariant under translations along the null generators of the null hypersurface. This condition is of course satisfied by Killing horizons of stationary black holes, but also by Rindler horizons, and more generally by other quasi-local notions of horizons such as non-expanding horizons and isolated horizons (see e.g. [8, 9] for reviews).

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In all these cases, we can generate an infinite number of physically distinct shells on this hypersurface, and we will refer to these generically as *Horizon Shells*. These shells are parametrised by one (essentially) arbitrary function on the horizon, which arises as an arbitrary coordinate transformation of the null coordinate v on the hypersurface (with coordinates (v, x^A)),

$$v \to F(v, x^A) \tag{1.1}$$

(extended to a suitable coordinate transformation off the shell on one side of the shell).[1] The resulting shells will in general carry a (null) matter energy momentum tensor (composed of energy density, energy currents and pressure), as well as impulsive gravitational waves travelling along the shell, and we determine and analyse these in some detail.

In particular, these horizon shells give rise to a rich *classical* structure at the horizon of a Schwarzschild black hole, and can be considered as significant generalisations of the Dray 't Hooft null shell [5]. As all of them are non-singular on the shell (i.e. do not have any point particle singularities), they also provide one with a wide array of smoothed out versions of Dray 't Hooft impulsive gravitational waves [11]. Given the recent interest in these configurations in the context of holography and scattering from black hole horizons (see e.g. [12, 13]), it is perhaps of some interest to have these new shell solutions at one's disposal.

A class of soldering transformations that may be of particular interest, especially in light of the observation in [14] that black holes must carry supertranslation hair, and the subsequent Hawking-Perry-Strominger proposal [15, 16], are the horizon analogues of BMS supertranslations at $\mathcal{I}^+$, of the form

$$v \to v + T(x^A) \; . \tag{1.2}$$

In particular, we find that for shells generated by supertranslations of the Kruskal coordinate V, the conserved energy $E[T]$ of the resulting shell is of the form (6.31)

$$E[T] = \frac{1}{8\pi} \int_{S^2} T \tag{1.3}$$

(there is also a simple, but slightly different, expression (7.10) for the conserved energy of shells generated by supertranslations of the advanced coordinate v). This expression bears a tantalising similarity to the standard expression for BMS supertranslation charges at $\mathcal{I}^+$ (see e.g. [17–19]) and to an analogous expression for a near-horizon BMS supercharge proposed recently in [20], and therefore suggests a direct relation between BMS supertranslation hair of a black hole and properties of the horizon shell.

Since (modulo isometries) there is a one-to-one correspondence between soldering transformations and null shells, we can think of the shell as a faithful bookkeeping device for BMS-like and more general soldering transformations. We therefore propose to interpret the abstract BMS charges in the context of horizon BMS transformations concretely as conserved physical charges (the energy, say) of the horizon shell that is (or could be) generated by the corresponding soldering transformation.

[1] Transformations of the form (1.1) have arisen previously in the discussion of asymptotic symmetries of Killing horizons in [10]. We thank Andy Strominger for bringing this reference to our attention.

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Since we were led to discover these structures from a systematic analysis of soldering transformations, and not from attempting to transfer structures from $\mathcal{I}^+$ to the horizon, there are some differences in perspective. For instance, we extend the soldering transformation from the horizon to a coordinate transformation inside the horizon rather than to the exterior (as would have been more natural in terms of the BMS perspective). The two procedures are of course completely equivalent.

Moreover, we require no notion of asymptotic symmetry but consider the most general transformation maintaining only the continuity of the induced metric. It is straightforward, however, to specialise our construction to whatever notion of asymptotic symmetry at the horizon is the relevant one in the case at hand. In particular, it is possible that not all the soldering transformations (1.1) allowed by the general construction are actually relevant in a specific context (like that of horizon hairs created by physical perturbations of the black hole metric, say), but only a specific subset of them, like the above supertranslations (1.2).

Finally, although there are many discussions of the role of conformal invariance in the physics of event horizons, we see no place for conformal transformations that are non-trivial on the horizon from our shell perspective.

In section 2 we analyse the soldering freedom in the Israel junction condition, and in section 3 we phrase these results in the language of Carrollian manifolds. Since we use the framework of continuous coordinates (coordinates in which all of the components of the metric are continuous across the shell, not only those of the induced metric) to derive the physical properties of the shell, in section 4 we explain how to lift soldering transformations off the shell in such a way that this continuity condition is satisfied. In section 5, we then obtain the general expressions for the energy-momentum tensor and impulsive gravitational wave components on the shell and discuss the corresponding conservation laws. In section 6, we present in an elementary fashion our main example, namely the general horizon shell on the horizon $U = 0$ of the Kruskal geometry, determine and analyse in detail its physical properties and study various special cases. Section 7 concludes with a brief discussion of the same example in Eddington-Finkelstein coordinates.

2 Soldering freedom and horizon shells

The general construction of null shells in general relativity [1–4] involves a matching of two manifolds with boundary, $\mathcal{V}^+$ to the future of a null hypersurface $\mathcal{N}^+$ and $\mathcal{V}^-$ to the past of $\mathcal{N}^-$ to each other across a common null boundary $\mathcal{N}$. Each of the manifolds $\mathcal{V}^\pm$ and $\mathcal{N}$ respectively have independent coordinate charts $x^\mu_\pm$ for $\mu = 0, 1, 2, 3$, and y^a for $a = 0, 1, 2$ and $\mathcal{V}^\pm$ carry metrics $g^\pm_{\mu\nu}$.

The basic requirement for the matching of the future and past geometries across the null hypersurface $\mathcal{N}$ is the (Israel) *junction condition* that the induced metrics

$$g^\pm_{ab} = g^\pm_{\mu\nu}|_{\mathcal{N}^\pm} \frac{\partial x^\mu_\pm}{\partial y^a} \frac{\partial x^\nu_\pm}{\partial y^b} \tag{2.1}$$

on the future and past boundaries are isometric. It is common to write this condition as

$$[g_{ab}] \equiv g^+_{ab} - g^-_{ab} = 0\ , \tag{2.2}$$

expressing the statement that there is no jump in the induced metric across the shell. This condition ensures on the one hand that the two boundaries can be identified with the hypersurface $\mathcal{N}$ with a unique metric, and on the other hand this construction leads to solutions of the Einstein equations that can be interpreted as describing a shell of matter and/or gravitational radiation separating (or joining) the two regions.

The general formalism of [1]–[4] then shows how to derive the intrinsic properties of the shell, independently of any coordinate choices. We will return to this issue later on. First of all, however, we want to ask and address the question whether there is any freedom in this procedure to "solder" the two geometries together or if it is completely specified by specifying the two geometries (satisfying the junction conditions) on the two sides.

For a generic hypersurface, the metric induced on the hypersurface will be a function of the way the hypersurface is embedded in the surrounding space-time. Thus, for generic hypersurfaces the interior and exterior space-times are soldered together in an essentially unique way dictated by the embedding (we will see this explicitly below, also for non-null shells). However, it was already pointed out in [1] by way of example that for certain null shells like the Killing horizons of static black holes, there is considerable freedom in how the two geometries are attached, allowing one to slide one of the manifolds independently along the null (isometry) direction on $\mathcal{N}$ before soldering. We will now investigate this question more sytematically.

In order to (significantly) simplify the analysis, as well as the calculations for specific examples later on in this paper, we will now introduce a preferred class of coordinate systems y^a on the shell $\mathcal{N}$, and also a corresponding class of space-time coordinates x^α in a neighbourhood of $\mathcal{N}$ (details of this rather standard construction can be found e.g. in [1]–[4]).

1. We introduce a coordinate system $y^a = (v, y^A)$ on $\mathcal{N}$ adapted to the fact that any null hypersurface is generated by null geodesics (the integral curves of any null normal n of $\mathcal{N}$), i.e. v is a parameter along the null geodesics related to the choice of null normal n by $n = \partial_v$, and the remaining spatial coordinates y^A are used to label the individual null geodesics. In these coordinates the induced degenerate metric on $\mathcal{N}$ has the form
$$g_{ab}n^b = g_{av} = 0 \quad \Rightarrow \quad g_{ab}(y)dy^a dy^b = g_{AB}(v, y^C)dy^A dy^B \ , \tag{2.3}$$
the conditions $g_{vv} = g_{vA} = 0$ expressing the fact that ∂_v is null and normal to $\mathcal{N}$.

2. We introduce a coordinate system x^α in a 2-sided neighbourhood of the shell $\mathcal{N}$ (e.g. via a null analogue of the construction of Gaussian normal coordinates) such that in this coordinate system all the components $g^{\pm}_{\alpha\beta}$ of the metric, not just those contributing to the induced metric, are continuous across $\mathcal{N}$,
$$[g_{\alpha\beta}] = 0 \ . \tag{2.4}$$

We should add here that typically such a coordinate system is only used as an intermediate auxiliary device in the literature, and is generally considered to be somewhat (or grossly [3]) impractical for actual calculations. However, for the applications of the formalism that we are interested in, the construction of such a coordinate system in a small neighbourhood of the shell turns out to be completely straightforward, since we start with a space-time metric with no shell (or, equivalently, philosophical questions aside, with an empty shell), which obviously already comes with its continuous coordinate system.

Anyway, given such a coordinate system the null boundary hypersurface $\mathcal{N}$ can then be described by an equation $\Phi(x) = 0$ where $\Phi(x)$ is a smooth function such that $\Phi > 0$ to the future $\mathcal{V}^+$ of $\mathcal{N}$, and $\Phi < 0$ on $\mathcal{V}^-$. It is then natural and convenient to choose one of the coordinates x^α to be proportional to $\Phi(x)$, so we set $x^\alpha = (u, x^a)$, with

$$\Phi(x) = \lambda u \tag{2.5}$$

(for some conveniently chosen constant λ), and thus $\mathcal{N}$ is simply given by $u = 0$. Finally, the two coordinate systems are linked by the choice that on $\mathcal{N}$ one has

$$x^a|_{\mathcal{N}} = y^a \ . \tag{2.6}$$

(and this is the only slightly non-standard but very natural and convenient choice that we make). Therefore our coordinates are

$$(x^\alpha) = (u, x^a) = \left(u, v, y^A\right) \ . \tag{2.7}$$

More on the choice of defining function $\Phi(x)$ and our conventions regarding its relation with the normal vector n can be found in section 5.

Equipped with this, we can now return to the question raised above regarding the uniqueness of the soldering procedure. Since a geometry is determined by a metric up to coordinate transformations, one way to approach this question is to enquire if there is the freedom to perform coordinate transformations on one side, say on $\mathcal{V}^+$, while maintaining the fundamental junction condition $[g_{ab}] = 0$.[2] Since on $\mathcal{N}$ we have identified the coordinates x^a with the intrinsic coordinates $y^a = (v, y^A)$, this amounts to asking under which coordinate transformations of the y^a the induced metric g_{ab} remains invariant. Infinitesimally

$$L_Z g_{ab} = Z^c \partial_c g_{ab} + (\partial_a Z^c) g_{cb} + (\partial_b Z^c) g_{ac} = 0 \ . \tag{2.8}$$

For a *timelike* (or spacelike) shell, to which these considerations up to this point would also apply, this is just the Killing equation, and therefore this argument shows that (as is well known from other perspectives) in this case the soldering is unique up to isometries of the induced metric (and these leave all physical quantities invariant).

For a *null* shell, with its degenerate metric, with $g_{av} = 0$ and g_{AB} non-degenerate, the situation is potentially more interesting. In this case, the above equation becomes

$$L_Z g_{ab} = Z^c \partial_c g_{ab} + \left(\partial_a Z^C\right) g_{Cb} + \left(\partial_b Z^C\right) g_{aC} = 0 \tag{2.9}$$

[2] One could of course obtain the same results by performing a coordinate transformation only on $\mathcal{V}^-$. These two different perspectives can then clearly be related via a smooth global coordinate transformation without altering the intrinsic properties of the shell.

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for a vector field $Z = Z^c\partial_c = Z^v\partial_v + Z^C\partial_C$ on $\mathcal{N}$. For $a = v$ or $b = v$ one finds the condition

$$L_Z g_{av} = 0 \quad \Rightarrow \quad \partial_v Z^A = 0 \,, \tag{2.10}$$

which rules out any v-dependent transformations of the spatial coordinates y^A. From the spatial components of (2.9) one finds

$$L_Z g_{AB} = 0 \quad \Rightarrow \quad Z^v\partial_v g_{AB} + Z^C\partial_C g_{AB} + \left(\partial_A Z^C\right) g_{CB} + \left(\partial_B Z^C\right) g_{AC} = 0 \,. \tag{2.11}$$

The corresponding group of allowed transformations does not depend on a particular choice of normalisation of the null generators, and is thus an object intrinsically associated to the null hypersurface and its metric. It is just the isometry group $\mathsf{Isom}(g_{ab})$ of the degenerate metric on $\mathcal{N}$, with Lie algebra

$$\mathfrak{isom}(\mathcal{N}, g_{ab}) = \{Z : \; L_Z g_{ab} = 0\} \,. \tag{2.12}$$

We will now take a closer look at the solutions of (2.11). For a generic v-dependence of g_{AB}, one only has rigid (i.e. v-independent) isometries of the metric g_{AB}, with $Z^v = 0$, i.e.

$$\text{generically:} \quad \mathsf{Isom}(\mathcal{N}, g_{ab}) = \mathsf{Isom}(g_{AB}) \tag{2.13}$$

(and as in the case of timelike shells there is then an essentially unique soldering).

While there are some special cases in which a v-dependent metric can possess non-trivial soldering transformations, as in the example of the Nutku-Penrose construction of impulsive gravitational waves on the light cone [21] (cf. also section 1.2 of [2] or the light cone example in [6, 7]), there is an especially interesting case that we will focus on here, in which the soldering group is not only non-trivial but actually infinite dimensional. This happens when the metric is independent of v, i.e. translation invariant along the null generators of $\mathcal{N}$,

$$n^c\partial_c g_{ab} = 0 \quad \Leftrightarrow \quad \partial_v g_{AB} = 0 \,. \tag{2.14}$$

In particular, this includes the *Killing Horizons* of stationary black holes and *Rindler Horizons* (with their boost Killing vector). More generally, since our considerations involve only $\mathcal{N}$ itself, this condition, which implies that the null congruence on $\mathcal{N}$ has zero expansion and shear, is satisfied by arbitrary *Non-expanding Horizons*, a precursor to *(Weakly) Isolated Horizons* (see e.g. [8, 9] for reviews). In the current context of null shells, we will therefore refer to null hypersurfaces with a metric satisfying $\partial_v g_{ab} = 0$ or $\partial_v g_{AB} = 0$ as *Horizon Shells*.

In this case the component Z^v is completely unconstrained, and corresponds to the freedom to perform arbitrary coordinate transformations of v on $\mathcal{N}$,

$$\partial_v g_{AB} = 0 \quad \Rightarrow \quad v \to F\left(v, y^A\right) \quad \text{allowed} \tag{2.15}$$

(here and in the following it is understood that the function F is such that $(v, y^A) \to (F, y^A)$ is a legitimate orientation preserving coordinate transformation, i.e. that F satisfies $\partial_v F > 0$). Because of the presence of an arbitrary function F in the coordinate transformation, the isometry group $\mathsf{Isom}(\mathcal{N}, g_{ab})$ is infinite dimensional, and factorises as the semi-direct product

$$\text{Horizon Shells}: \quad \mathsf{Isom}(\mathcal{N}, g_{ab}) = \mathsf{Isom}(g_{AB}) \ltimes \mathsf{Sol}(\mathcal{N}) \ , \tag{2.16}$$

where the infinite dimensional soldering group $\mathsf{Sol}(\mathcal{N})$ (of non-trivial soldering transformations) is the group of coordinate transformations (2.15),

$$\mathsf{Sol}(\mathcal{N}, g) = \left\{ v \to F\left(v, y^A\right) \right\} \ . \tag{2.17}$$

As a consequence there are e.g. an infinite number of ways to glue two black holes geometries together (the inside horizon region of one to the exterior region of the other) provided only that the Israel junction condition is satisfied. In sections 6 and 7 we will analyse in detail the simplest example exhibiting this phenomenon, namely the soldering of two equal mass Schwarzschild metrics along their horizon.

Let us also analyse the effect of the transformations generated by Z on the normal null generator n^a of $\mathcal{N}$ for a general null shell. Since the metric is invariant by the condition (2.9), and n^a spans the kernel of the metric, it is clear a priori that L_Z preserves the direction of n^a, $L_Z n^a \sim n^a$,

$$\begin{aligned} g_{ab}n^b = 0 \quad &\Rightarrow \quad 0 = L_Z\left(g_{ab}n^b\right) = (L_Z g_{ab})n^b + g_{ab}L_Z n^b = g_{ab}L_Z n^b \\ &\Rightarrow \quad L_Z n^a \sim n^a \ . \end{aligned} \tag{2.18}$$

This can also be seen explicitly from $n = \partial_v$, i.e. $n^a = \delta^a_v$, and (2.10),

$$L_Z n^a = Z^b \partial_b n^a - n^b \partial_b Z^a = -\partial_v Z^a = -(\partial_v Z^v)\delta^a_v = -(\partial_v Z^v) n^a \ . \tag{2.19}$$

If one further restricts the transformations to those that strictly preserve the normal n^a, then the allowed coordinate transformations on the shell are restricted to $\partial_v Z^v = 0$. In the case of horizon shells, this restricts the soldering transformations (2.15) to

$$L_Z n^a = 0 \quad \Rightarrow \quad \partial_v Z^v = 0 \quad \Rightarrow \quad v \to v + F\left(y^A\right) \quad \text{allowed} \ . \tag{2.20}$$

Thus even when there is a preferred (Killing, say) null generator of the horizon shell, the soldering group preserving this structure is still infinite dimensional.

3 Soldering group, Carrollian manifolds and BMS transformations

Interestingly the above considerations, motivated by the question of the freedom in soldering null shells, are closely related to recent investigations of the (ultra-relativistic) Carroll group and various other symmetry groups of Carrollian manifolds and Carrollian structures and their relation with BMS supertranslation symmetries (cf. in particular [6, 7]).

A *Carrollian manifold* is by definition (we adopt the "weak" definition of [6, 7]) a manifold equipped with a degenerate non-negative metric whose kernel is everywhere 1-dimensional (and is thus spanned by a nowhere vanishing vector field). Clearly, according to this definition any null hypersurface of a Lorentzian (pseudo-Riemannian) space-

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time defines a Carrollian manifold, the Carroll structure being encoded in the triplet $(\mathcal{N}, g_{ab}, n^a) \equiv (\mathcal{N}, g, n)$.[3]

Given a Carrollian structure, one can then analyse various notions of symmetry groups preserving (in a suitable sense) such a structure. In particular, the group of transformations preserving both g_{ab} and n^a could be called the isometry group $\mathsf{Isom}(\mathcal{N}, g, n)$. Its Lie algebra is generated by the vector fields Z on $\mathcal{N}$ satisfying

$$\mathfrak{isom}(\mathcal{N}, g, n) = \{Z : \ L_Z g = L_Z n = 0\} \ . \tag{3.1}$$

As we have seen, in the case of horizon shells (as defined above) this isometry group is infinite dimensional, due to the presence of the transformations (2.20). By extrapolation from the corresponding terminology for future null infinity $\mathcal{I}^+$, which is a natural example of a Carrollian manifold, these transformations are called

$$\text{BMS supertranslations :} \quad v \to v + F(y^A) \ , \tag{3.2}$$

and form an infinite dimensional Abelian subgroup of the isometry group of the Carrollian structure on such a null surface.

Note that, via the identification $n = \partial_v$, this definition of horizon shell BMS transformations depends on the choice of null normal. For example, unlike at $\mathcal{I}^+$, at the horizon of a static black hole supertranslations of the Killing parameter (Eddington-Finkelstein advanced v) are not the same as supertranslations of the affine parameter (Kruskal V).

In passing we also note that for horizon shells $(\mathcal{N}, g, n)$ also defines a Carrollian structure in the strong sense of [6, 7], which requires the existence of a symmetric affine connection ∇ compatible with $(\mathcal{N}, g, n)$, i.e.

$$\nabla_c g_{ab} = 0 \ , \qquad \nabla_c n^a = 0 \ . \tag{3.3}$$

Indeed, it is easy to see that any symmetric connection $\Gamma^a_{\ bc}$ with

$$\Gamma^A_{\ BC} = \Gamma^A_{\ BC}(g) \ , \qquad \Gamma^a_{\ bv} = 0 \ , \qquad \Gamma^v_{\ AB} \quad \text{arbitrary} \tag{3.4}$$

($\Gamma^A_{\ BC}(g)$ are the components of the Levi-Civita connection of the spatial metric g_{AB}) satisfies this condition. As the existence of such a connection does not appear to play a significant role in the context we are interested in, there is also no reason to restrict the soldering transformations to the group of transformations that preserve the connection (which would be finite-dimensional).

In the Carrollian setting, the more general allowed soldering transformations (2.15) appear as a subgroup of what (again by extrapolation from $\mathcal{I}^+$) are called *Newman-Unti transformations*. Due to its $\mathcal{I}^+$ pedigree, the Newman-Unti group also contains conformal isometries. In the present context of the soldering of horizon shells, conformal isometries are not allowed (as they fail to satisfy (2.11)), and the relevant group is the group $\mathsf{Sol}(N, g)$ (2.17) of soldering transformations.

[3]For the converse question, how to embed a given Carrollian manifold as a null hypersurface into an ambient space-time, see [22, 23].

4 Off-horizon shell extension of the soldering transformations

In keeping with the framework we have adopted for discussing shells, we now need to understand how to extend the soldering transformations off the shell $\mathcal{N}$ in such a way that the entire metric remains continuous across the shell,

$$[g_{\alpha\beta}] = 0 \; . \tag{4.1}$$

We thus asume that we have a "seed" metric which satisfies (4.1) (in the example of section 6 this is simply the Schwarzschild metric in Kruskal coordinates, with an empty shell on the horizon $U = 0$).

We now want to lift the soldering transformation generated by $Z = Z^v \partial_v$ on $\mathcal{N}$ to a coordinate transformation in an infinitesimal neighbourhood of one side of the shell, say $\mathcal{V}^+$, in such a way that the continuity of the metric (4.1) is maintained. I.e. we extend the vector field off $\mathcal{N}$ as

$$Z_+ = Z^v \partial_v + u z^\alpha \partial_\alpha \tag{4.2}$$

and impose the requirement that

$$L_{Z_+} g_{\alpha\beta}|_{\mathcal{N}} = 0 \; . \tag{4.3}$$

Because on $\mathcal{N}$ the vector field generates a soldering transformation, the conditions

$$L_{Z_+} g_{ab}|_{\mathcal{N}} = 0 \tag{4.4}$$

are identically satisfied. Indeed, since

$$Z_+|_{\mathcal{N}} = Z^v \partial_v \; , \qquad \partial_a Z_+|_{\mathcal{N}} = (\partial_a Z^v)\partial_v \; , \tag{4.5}$$

one has

$$\begin{aligned} L_{Z_+} g_{ab}|_{\mathcal{N}} &= (Z^v \partial_v g_{ab} + \partial_a Z^\alpha g_{\alpha b} + \partial_b Z^\alpha g_{a\alpha})\,|_{\mathcal{N}} \\ &= (Z^v \partial_v g_{ab} + \partial_a Z^v g_{vb} + \partial_b Z^v g_{av})\,|_{\mathcal{N}} = 0 \end{aligned} \tag{4.6}$$

because on the null horizon shell $\partial_v g_{ab} = g_{av} = 0$. It thus remains to impose the conditions

$$L_{Z_+} g_{u\beta}|_{\mathcal{N}} = 0 \; . \tag{4.7}$$

These are linear equations for the coefficients z^α,

$$\begin{aligned} \beta = u: &\quad Z^v \partial_v g_{uu} + 2 z^\alpha g_{\alpha u} = 0 \\ \beta = v: &\quad Z^v \partial_v g_{uv} + (z^u + (\partial_v Z^v)) g_{uv} = 0 \\ \beta = B: &\quad Z^v \partial_v g_{uB} + z^\alpha g_{\alpha B} + (\partial_B Z^v) g_{uv} = 0 \end{aligned} \tag{4.8}$$

(here the restriction to $\mathcal{N}$ is implied).

In particular, if we now restrict to metrics with

$$g_{uu} = g_{uA} = 0 \; , \qquad (\partial_v g_{uv})|_{\mathcal{N}} = \partial_v (g_{uv}|_{\mathcal{N}}) = 0 \; , \tag{4.9}$$

(typical examples would be spherically symmetric metrics in null or double-null coordinates like the Schwarzschild metric in Eddington-Finkelstein or Kruskal coordinates), the solutions to the above equations are

$$z^v = 0\ , \qquad z^u = -\partial_v Z^v\ , \qquad z^A = -g_{uv} g^{AB} \partial_B Z^v\ . \tag{4.10}$$

E.g. for the Schwarzschild metric in Kruskal coordinates (with $u = U, v = V$) one has

$$z^V = 0\ , \qquad z^U = -\partial_V Z^V\ , \qquad z^A = (2/e)\sigma^{AB} \partial_B Z^V\ , \tag{4.11}$$

with σ_{AB} the standard metric on the unit 2-sphere. Defining the function $\omega(v, \theta, \phi)$ by

$$Z^V = V\omega(V, \theta, \phi)\ , \tag{4.12}$$

the generator (4.2) of the off-shell extension of the soldering transformation can be rewritten in the suggestive form

$$Z_+ = \omega\,(V\partial_V - U\partial_U) + U\left(z^A\partial_A - V(\partial_V\omega)\partial_U\right) \tag{4.13}$$

of a "local Killing transformation" (with respect to the Kruskal Killing vector $\sim V\partial_V - U\partial_U$ and with local coefficient $\omega(V, \theta, \phi)$), as was first found in the discussion of the asymptotic symmetries of Killing horizons in [10]. That the soldering transformation can be written as a local Killing transformation is even more transparent in Eddington-Finkelstein coordinates, with Killing vector ∂_v, where (4.2) has this form on the nose.

In principle this can now be exponentiated to find the exact coordinate transformation to linear order in u. In practice, however, this is a bit tedious, and in the following we will obtain this transformation directly, starting from the ansatz

$$v_+ = F(x^a) + uA(x^a)\ , \qquad x^A_+ = x^A + uB^A(x^a)\ , \qquad u_+ = uC(x^a) \tag{4.14}$$

(the linear term uA in v_+ will be generated by exponentiation of (4.10)), and demanding continuity of the metric, $[g_{\alpha\beta}] = 0$. Continuity of $g_{u\alpha}$ then determines the functions $B^\alpha = (A, C, B^A)$.

5 From soldering to the physical properties of the shell

Assuming that we have successfully implemented the steps of finding a non-trivial soldering transformation and its off-shell lift, we can now determine (pretty much read off) the physical properties of the shell.

We just need to be slightly more specific about our conventions regarding the choice of normal vector. Recall from section 2 that the shell is given by the equation $\Phi(x) = \lambda u = 0$. A null normal (and tangent) to $\mathcal{N}$ is then given by

$$n_\alpha = -\partial_\alpha \Phi|_{\mathcal{N}} = -\lambda \partial_\alpha u|_{\mathcal{N}}\ . \tag{5.1}$$

Here the sign is chosen such that n^α is future pointing when Φ increases towards the future, and we absorb the freedom to multiply n^α by an arbitrary non-vanishing function on $\mathcal{N}$

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into the freedom in the choice of Φ. Since on $\mathcal{N}$ we had already chosen $n = \partial_v$, this correlates the choice of Φ with the choice of coordinate v,

$$n^\alpha \partial_\alpha = n^a \partial_a = \partial_v \ . \tag{5.2}$$

As the metric is continuous across the shell in the chosen coordinates, so are all its tangential derivatives. In order to be able to take derivatives in the direction transverse to the shell, we need to introduce an auxiliary vector field N that is transverse to $\mathcal{N}$, i.e. $n.N \neq 0$, and continuous across $\mathcal{N}$, $[N] = 0$. A convenient choice is

$$N = \lambda^{-1}\partial_u \quad \Rightarrow \quad N^\alpha n_\alpha = -1 \ . \tag{5.3}$$

It then turns out that the complete information about the intrinsic physical properties of the shell is encoded in the first transverse derivative of the tangential components of the metric (the choice to make the non-tangential components $g_{\alpha u}$ continuous was just a gauge choice and has no influence on the physics). Therefore the basic shell-intrinsic tensor that contains all information about the properties of the shell is

$$\gamma_{ab} = N^\alpha[\partial_\alpha g_{ab}] = \lambda^{-1}[\partial_u g_{ab}] = [L_N g_{ab}] \ . \tag{5.4}$$

A simple way to determine the γ_{ab} is to expand the tangential components of the interior and exterior metrics to linear order in u,

$$ds^2 = \left(g^{(0)}_{ab} + u g^{(1)\pm}_{ab} + \dots\right) dx^a dx^b \ , \tag{5.5}$$

and to then read off the γ_{ab} from

$$\gamma_{ab} = \lambda^{-1}\left[g^{(1)}_{ab}\right] \ . \tag{5.6}$$

Starting from the soldering transformation $v \to F(v, y^A)$, the γ_{ab} are then given explicitly in terms of F and its 1st and 2nd derivatives,

$$\gamma_{ab} = \gamma_{ab}(F, \partial_a F, \partial_a\partial_b F) \ . \tag{5.7}$$

In terms of the γ_{ab}, the intrinsic energy-momentum tensor of the shell is in turn given by

$$16\pi S^{ab} = -\gamma^* n^a n^b - \gamma^\dagger g^{ab}_* + 2\gamma^{(a}n^{b)} \tag{5.8}$$

where $g^{ab}_* = \delta^a{}_A \delta^b{}_B g^{AB}$ is the inverse of the spatial (non-degenerate) part of the hypersurface metric and

$$\gamma^* = g^{ab}_*\gamma_{ab} = g^{AB}\gamma_{AB} \ , \qquad \gamma_a = \gamma_{ab}n^b \ , \qquad \gamma^a = g^{ab}_*\gamma_b \ , \qquad \gamma^\dagger = \gamma_{ab}n^a n^b = \gamma_a n^a \ . \tag{5.9}$$

The components of S^{ab} can be interpreted as surface energy density μ, pressure p and energy currents j^A, where

$$\mu = -\frac{1}{16\pi}\gamma^* \ , \qquad j^A = +\frac{1}{16\pi}\gamma^A \ , \qquad p = -\frac{1}{16\pi}\gamma^\dagger \ . \tag{5.10}$$

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In particular, the pressure is related to the jump in the surface gravity or inaffinity κ, defined by

$$n^\beta \nabla_\beta n^\alpha = \kappa n^\alpha \,. \tag{5.11}$$

Indeed, taking the scalar product with the transverse vector N and using $N.n = -1$ and the definition of γ_{ab} (5.4), one finds

$$\kappa = -n^\beta N_\alpha \nabla_\beta n^\alpha = (\nabla_\beta N_\alpha) n^\beta n^\alpha = \frac{1}{2}(L_N g_{\alpha\beta}) n^\alpha n^\beta = \frac{1}{2}(L_N g_{ab}) n^a n^b \,, \tag{5.12}$$

and therefore

$$[\kappa] = \frac{1}{2}\gamma_{ab} n^a n^b = \frac{1}{2}\gamma^\dagger \,. \tag{5.13}$$

Note that there is an ambiguity in the operational interpretation of the components of

$$S^{ab} = \mu n^a n^b + j^a n^b + j^b n^a + p g_*^{ab} \tag{5.14}$$

due to absence of a rest-frame on the null shell $\mathcal{N}$. As a consequence, different observers upon crossing the shell will measure rescalings of μ, p and j^A as described in detail in section 3.11 of [4].

For a general null shell the surface energy-momentum tensor has only these 4 independent components (in contrast to a timelike shell, which has 6), while γ_{ab} has 6 components. Indeed it is evident from the expressions (5.9) that the transverse traceless components $\hat{\gamma}_{ab}$ of γ_{ab}, characterised by

$$\hat{\gamma}_{ab} n^b = 0 \,, \qquad g_*^{ab}\hat{\gamma}_{ab} = 0 \,, \tag{5.15}$$

do not contribute to the matter content on the shell encoded in S^{ab}. Instead, these components, which can be extracted from γ_{ab} according to

$$\hat{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\gamma^* g_{ab} + 2\gamma_{(a}N_{b)} + \left(N_a N_b - \frac{1}{2}N.N g_{ab}\right)\gamma^\dagger \,, \tag{5.16}$$

contribute to the Weyl tensor on the shell and describe the 2 polarisation states of an impulsive gravitational wave travelling along the shell. For more details on this gravitational wave component in general see e.g. section 2.3 of [2].

Finally, we note that the shell energy-momentum tensor S^{ab} satisfies certain conservation laws derived in [1] (their derivation requires some care, due to the degeneracy of the metric, and because S^{ab} is only defined on the shell). In the absence of bulk matter these are

$$N_a \left(\partial_b + \tilde{\Gamma}_b\right) S^{ab} - S^{ab}\tilde{\mathcal{K}}_{ab} = 0 \tag{5.17}$$

and

$$S^b_{a;b} = \left(\partial_b + \tilde{\Gamma}_b\right) S^b_a - \frac{1}{2}S^{bc}\partial_a g_{bc} = 0 \,. \tag{5.18}$$

Here

$$S^b_a \equiv g_{ac} S^{cb} = \frac{1}{16\pi}\left(n^b \gamma_a - \delta^b_a \gamma^\dagger\right) \,, \tag{5.19}$$

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a tilde over a quantity denotes an average value of the quantities from the two sides of the shell, $\tilde{\Gamma}_b$ is the average of the null surface counterpart of the contracted Christoffel symbol, defined (in the coordinates that we have used here) as

$$\Gamma^{\pm}_b = \nabla^{\pm}_\mu \delta^\mu_b = \Gamma^{\pm\mu}_{\ \mu b} \, , \tag{5.20}$$

and finally $\mathcal{K}_{ab}$ is the "transverse extrinsic curvature", satisfying $\gamma_{ab} = 2[\mathcal{K}_{ab}]$ and given in our coordinates simply by

$$\mathcal{K}^{\pm}_{ab} = (2\lambda)^{-1} g^{\pm(1)}_{ab} \, . \tag{5.21}$$

It may be a bit puzzling that one obtains 4 equations for S^{ab}. We will see that (5.17) and the spatial components of (5.18) give rise to the 3 expected conservation laws for S^{ab}. The v-component of (5.18) (equivalently, the contraction of (5.18) with n^a), however, is simply a geometric identity which (as such) is identically satisfied. Temporarily including also bulk matter, this contracted equation reads

$$n^a S^b_{a;b} = \left[T_{\alpha\beta} n^\alpha n^\beta\right] = \left[T_{ab} n^a n^b\right] \, . \tag{5.22}$$

Noting that $n^a S^b_a = 0$, we find that the left-hand side is simply

$$n^a S^b_{a;b} = -\frac{1}{2} n^a S^{bc} \partial_a g_{bc} = -\frac{1}{2} S^{BC} \partial_v g_{BC} = -\frac{1}{2} p g^{BC} \partial_v g_{BC} = \frac{1}{8\pi}[\kappa]\theta \tag{5.23}$$

where θ is the expansion of the null congruence on $\mathcal{N}$. On the other hand, the Raychaudhuri equation for this congruence reads (upon using the Einstein equations)

$$\partial_v \theta + \frac{1}{2}\theta^2 + \sigma^{ab}\sigma_{ab} = \kappa\theta - 8\pi T_{ab} n^a n^b \tag{5.24}$$

(with σ_{ab} the spatial shear tensor). Since the left-hand side of this equation depends only on the intrinsic geometry of $\mathcal{N}$, its jump is zero, and therefore one finds [4]

$$[\kappa]\theta = 8\pi \left[T_{ab} n^a n^b\right] \tag{5.25}$$

as a geometric identity, and comparison with the above shows that this is precisely the content of (5.22). In the context of horizon shells, with no bulk matter, this equation is identically satisfied (without any constraint on the pressure) because the expansion $\theta = 0$. The remaining 3 equations can be seen to be satisfied for the S^{ab} that we derive in the examples.[4]

6 Schwarzschild horizon shell in Kruskal coordinates

To illustrate the above construction, we will now look in detail at the Schwarzschild (Kruskal) horizon shell. Thus we start with the Kruskal metric without a shell as our seed metric, and then we generate non-trivial shells on the horizon between two equal mass black hole metrics via soldering transformations and determine the physical properties of the shell.

[4]We should add that the fact that we obtain 4 valid equations without any restrictions on the "type" of the null hypersurface appears to contradict remarks in section 3.5 of [2] that this should not be possible.

6.1 General construction

In Kruskal coordinates the Schwarzschild metric is

$$ds^2 = -2G(r)dUdV + r^2 d\Omega^2 \ , \tag{6.1}$$

with

$$G(r) = \frac{16m^3}{r}\mathrm{e}^{-r/2m} \ , \qquad UV = -\left(\frac{r}{2m} - 1\right)\mathrm{e}^{r/2m} \ . \tag{6.2}$$

We choose the horizon shell $\mathcal{N}$ to be the Kruskal horizon $U = 0$. As discussed in the previous sections, we then have a large soldering freedom

$$V \to F(V, \theta, \phi) \tag{6.3}$$

as a consequence of the fact that the null generators are orbits of an intrinsic isometry of the induced hypersurface metric, but we can also quickly rederive this from scratch here. To that end, we introduce coordinates $(U_\pm, V_\pm, \theta_\pm, \phi_\pm)$ on $\mathcal{V}^+$ $(U \geq 0)$ and $\mathcal{V}^-$ $(U \leq 0)$ respectively, so that we have

$$ds^2_\pm = -2G(r_\pm)dU_\pm dV_\pm + r^2_\pm \left(d\theta^2_\pm + \sin^2\theta_\pm d\phi^2_\pm\right) \quad \text{on } \ \mathcal{V}^\pm \ . \tag{6.4}$$

Therefore, the metric induced on the horizon $\mathcal{N}$ at $U = 0 \to r = 2m$ is

$$ds^2_\pm|_\mathcal{N} = 4m^2(d\theta^2_\pm + \sin^2\theta_\pm d\phi^2_\pm) \ . \tag{6.5}$$

This induced metric is continuous across the shell (i.e. satisfies the Israel junction condition (2.2)) provided that we choose

$$\theta_+|_\mathcal{N} = \theta_-|_\mathcal{N} \ , \qquad \phi_+|_\mathcal{N} = \phi_-|_\mathcal{N} \tag{6.6}$$

(up to isometry rotations). However, as this metric does not depend on the coordinates $V_\pm$ (Killing horizon), we are free to choose any relation

$$V_+|_\mathcal{N} = F(V_-, \theta_-, \phi_-) \tag{6.7}$$

between the coordinates $V_\pm$ while maintaining the junction condition.[5] This is precisely the soldering freedom (6.3) mentioned above.

As continuous coordinates we now choose the Kruskal coordinates

$$x^\alpha = (U, x^a) = (U = U_-, V = V_-, \theta = \theta_-, \phi = \phi_-) \ , \tag{6.8}$$

and the coordinates on the shell are then (naturally) taken to be $y^a = x^a = (V, \theta, \phi)$. An obvious candidate for the defining function $\Phi(x)$ with $\mathcal{N} = \{x : \ \Phi(x) = 0\}$ is

$$\Phi(x) = \Phi(U) = \lambda U \tag{6.9}$$

[5] As already mentioned in section 2, we could of course equivalently perform this transformation on the exterior coordinate V_- instead of "hiding" it behind the horizon.

with a constant λ to be suitably chosen. As explained in section 5, this gives rise to a corresponding choice of null normal (5.1)

$$n_\alpha = -\partial_\alpha \Phi|_{\mathcal{N}} = -\lambda \partial_\alpha U|_{\mathcal{N}} \quad \Rightarrow \quad n = n^a \partial_a = (\lambda/G(2m))\partial_V \ . \tag{6.10}$$

In order to match with our choice of coordinates on $\mathcal{N}$, we choose $\lambda = G(2m)$, and thus n and the transverse null vector field N with $n.N = -1$ (5.3) are

$$n = \partial_V \ , \qquad N = \frac{1}{G(2m)}\partial_U \ . \tag{6.11}$$

In order to determine the physical properties of the shell generated by the above soldering transformation, we now extend the soldering transformation off the shell to a small neighbourhood of $\mathcal{N}$ in $\mathcal{V}^+$ such that all the components $g_{\alpha\beta}$ of the metric are continuous across the shell, not just those of the induced metric.

To the order in U necessary for the calculation of the energy-momentum tensor of the shell, the off-shell extension of the soldering transformation has the general form (4.14)

$$V_+ = F\left(V, x^A\right) + UA\left(V, x^A\right) \ , \qquad x_+^A = x^A + UB^A\left(V, x^A\right) \ , \qquad U_+ = UC\left(V, x^A\right) \tag{6.12}$$

(with $x^A = (\theta, \phi)$). Requiring continuity of the metric in Kruskal coordinates, specifically of the components $g_{U\alpha}$, determines

$$C = \frac{1}{F_V} \ , \qquad B^A = \frac{2}{e}\frac{\sigma^{AB}F_B}{F_V} \ , \qquad A = \frac{e}{4}F_V \sigma_{AB}B^A B^B \ , \tag{6.13}$$

with σ_{AB} the components of the metric on the unit 2-sphere, and $F_V = \partial_V F, F_B = \partial_B F$.

Infinitesimally, this coordinate transformation reduces precisely to the transformation given in (4.11), with A being generated as a higher order correction by exponentiation of (4.11). One can easily show that A can be set to zero by adding higher order U^2-terms to the transformations of the remaining coordinates (which has no effect on the properties of the shell itself).[6] We can and will therefore set A to zero for the calculation below.

To calculate the energy-momentum tensor, we follow the procedure described in section 5 and determine the γ_{ab} from the first-order expansion of the metric (5.6)

$$\gamma_{ab} = \frac{1}{G(2m)}\left[g_{ab}^{(1)}\right] \ . \tag{6.14}$$

While we only need the leading order term in $G(r)$, $G(2m) = 8m^2/e$, we need to expand r^2 and $\sin^2\theta_+$ to linear order in U. On $\mathcal{V}_-$ we just need

$$r(UV)^2 = 4m^2 - \frac{8m^2}{e}UV + \cdots \ , \tag{6.15}$$

and on $\mathcal{V}_+$ we have

$$\begin{aligned} r(U_+V_+)^2 &= 4m^2 - \frac{8m^2}{e}U_+V_+ + \cdots = 4m^2 - \frac{8m^2}{e}\frac{UF}{F_V} + \cdots \\ \sin^2\theta_+ &= \sin^2\theta + 2UB^\theta \sin\theta\cos\theta + \cdots \end{aligned} \tag{6.16}$$

[6] This is an alternative way to understand the "gauge invariance" of the space-time components $S^{\mu\nu}$ of the shell energy momentum tensor under certain transformations of the space-time $\gamma_{\mu\nu}$ discussed in [1].

Putting everything together we obtain

$$
\begin{aligned}
g^{(1)-}_{ab}dx^a dx^b &= -(8m^2/e)V\sigma_{AB}dx^A dx^B \\
g^{(1)+}_{ab}dx^a dx^b &= 8m^2\left(-(2/e)dC\,dF + \sigma_{AB}dx^A\left(dB^B - (F/eF_V)dx^B\right) + \sin\theta\cos\theta B^\theta d\phi^2\right)
\end{aligned}
\tag{6.17}
$$

Using the explicit expressions (6.13), one then finds

$$
\begin{aligned}
\gamma_{Va} &= 2\frac{\partial_V\partial_a F}{F_V} = 2\partial_a \log F_V \\
\gamma_{AB} &= 2\left(\frac{\nabla^{(2)}_A\partial_B F}{F_V} - \frac{1}{2}\sigma_{AB}\left(\frac{F}{F_V} - V\right)\right)\,,
\end{aligned}
\tag{6.18}
$$

where $\nabla^{(2)}$ is the 2-dimensional covariant derivative associated with σ_{AB}. The explicit expressions for the energy density, surface currents and pressure are then

$$
\begin{aligned}
\mu &= -\frac{1}{32m^2\pi F_V}\left(\Delta^{(2)}F - F + VF_V\right) \\
j^A &= \frac{1}{32m^2\pi}\sigma^{AB}\frac{F_{BV}}{F_V} \\
p &= -\frac{1}{8\pi}\frac{F_{VV}}{F_V}\,.
\end{aligned}
\tag{6.19}
$$

These quantities satisfy the conservation laws (5.17) and (5.18). We had already observed that the null component of (5.18) is identically satisfied. The conservation law (5.17) can be shown to be

$$
(\partial_V + \tilde{\kappa})\mu + \left(\nabla^{(2)}_A + \frac{1}{2}\gamma_{AV}\right)j^A + \frac{1}{4}\left(g^{AB}\gamma_{AB} - 4V\right)p = 0\,. \tag{6.20}
$$

Here $\tilde{\kappa}$ is the average value of the surface gravity (inaffinity) on the two sides. Since V is affine on $\mathcal{V}^-$, one has

$$
\tilde{\kappa} = \frac{1}{2}(\kappa^+ + \kappa^-) = \frac{1}{2}\kappa^+\,. \tag{6.21}
$$

Then it is straightforward to check that (6.20) is satisfied by the quantities in (6.19).

The spatial (angular) components of (5.18) become

$$
\partial_V\gamma_{AV} = \partial_A\gamma_{VV} \quad \Leftrightarrow \quad \partial_V j_A + \partial_A p = 0\,, \tag{6.22}
$$

which are obviously also identically satisfied by virtue of (6.18). Note also that the (gradient) currents have the additional property

$$
\partial_B j_A - \partial_A j_B = 0\,. \tag{6.23}
$$

Finally, the two remaining components $\hat{\gamma}_{ab}$ of γ_{ab} (5.16) give the 2 polarisations of an impulsive gravitational wave travelling along the shell, and we find in the above example that they are

$$
\begin{aligned}
\hat{\gamma}_{\theta\phi} &= \gamma_{\theta\phi} = 2\frac{\nabla^{(2)}_\theta\partial_\phi F}{F_V} \\
\hat{\gamma}_{\theta\theta} &= -\frac{1}{\sin^2\theta}\hat{\gamma}_{\phi\phi} = \frac{1}{2}\left(\gamma_{\theta\theta} - \frac{1}{\sin^2\theta}\gamma_{\phi\phi}\right) = \frac{2}{F_V}\left(\nabla^{(2)}_\theta\partial_\theta F - \frac{1}{\sin^2\theta}\nabla^{(2)}_\phi\partial_\phi F\right)
\end{aligned}
\tag{6.24}
$$

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6.2 Special cases

In order to get acquainted with these shells and their properties, we now specialise the above general construction in various ways.

- Dray 't Hooft Shell

  The simplest example is the Dray 't Hooft shell [5] which one obtains from a constant shift $V_+ = V + b$, with

$$\mu = \frac{b}{8\pi(2m)^2} \,, \qquad j_A = p = 0 \,. \tag{6.25}$$

  As γ_{AB} is pure trace for this constant shift, there is also no accompanying gravitational wave contribution in this case. Thus all the horizon shells we are considering that are generated by more general soldering transformations can be regarded as a broad class of generalisations of the Dray 't Hooft shell.

- Zero Pressure: $p = 0$

  From $p \sim \gamma_{VV} \sim F_{VV}$ it follows that

$$p = 0 \quad \Rightarrow \quad F(V, \theta, \phi) = A(\theta, \phi)V + B(\theta, \phi) \,. \tag{6.26}$$

  This result, an angle-dependent affine transformation of V, can also be understood from the fact that these are precisely the transformations that leave the inaffinity (surface gravity) κ invariant, so that $p \sim [\kappa] = 0$.

- Zero Pressure and Vanishing Currents: $p = j_A = 0$

  Since $j_A \sim \gamma_{VA} \sim F_{VA}$, we next deduce

$$j_A = 0 \quad \Rightarrow \quad F(v, \theta, \phi) = aV + B(\theta, \phi) \,. \tag{6.27}$$

- BMS Supertranslations of V

  From this we learn that, up to the irrelevant rescaling $V \to aV$ (which is part of the isometry $(V, U) \to (aV, a^{-1}U)$ of the Kruskal metric), BMS supertranslations

$$V \to F(V, \theta, \phi) = V + T(\theta, \phi) \tag{6.28}$$

  can be characterised as precisely those transformations that lead to a shell with zero pressure and currents. As we will see below, generically they will also generate accompanying impulsive gravitational waves travelling along the shell.

  The energy density for such a shell is

$$\mu = -\frac{1}{32m^2\pi}\left(\Delta^{(2)}T - T\right) \,, \tag{6.29}$$

  and correspondingly the conservation law (6.20) reduces to the evidently correct statement

$$\partial_V \mu = 0 \,. \tag{6.30}$$

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Thus we can define a conserved shell energy $E[T]$ by integrating μ over the spatial cross-section of the horizon. In this case, the 1st term in (6.29) does not contribute and we are left with

$$E[T] = \frac{1}{32\pi m^2} \int \sqrt{|g_{AB}|} d^2x \, T(\theta, \phi) = \frac{1}{8\pi} \int_{S^2} T(\theta, \phi) \ . \tag{6.31}$$

See the introduction for some discussions of this result.

- No Matter on the Shell: $S^{ab} = 0$

  With $F = aV + B$ one finds that the remaining angular components γ_{AB} of γ_{ab} are

$$\gamma_{AB} = (2/a) \left(\nabla^{(2)}_A \partial_B B - \frac{1}{2} \sigma_{AB} B \right) \ . \tag{6.32}$$

  Thus, the requirement that also the energy density $\mu \sim \sigma^{AB} \gamma_{AB}$ vanishes is

$$\mu = 0 \quad \Rightarrow \quad \Delta^{(2)} B = B \ . \tag{6.33}$$

  As there are no (non-singular) solutions to this equation (the eigenvalues of the Laplacian are $-\ell(\ell+1) \leq 0$), we conclude that

$$\mu = 0 \quad \Rightarrow \quad B = 0 \ , \tag{6.34}$$

  so that the only transformations leading to a shell with vanishing energy-momentum tensor are the constant rescalings $F(V) = aV$, corresponding to the scaling isometry in Kruskal coordinates already mentioned above.

- Nonexistence of Pure Impulsive Gravitational Waves

  With $B = 0$, not only the trace of γ_{AB} is zero, but evidently $\gamma_{AB} = 0$. Therefore in this Kruskal horizon shell example we have

$$S^{ab} = 0 \quad \Rightarrow \quad \gamma_{ab} = 0 \ . \tag{6.35}$$

  In particular, since the impulsive gravitational wave contributions are encoded in the transverse traceless part $\hat{\gamma}_{ab}$ of γ_{ab}, this means that in the case at hand there can be no pure impulsive gravitational waves without matter on the shell.

  This should be contrasted with the case of null hypersurfaces in Minkowski space [24] (cf. also section 2.4 of [2]) which, in the present setting, we can think of as Rindler horizons. In this case matter and gravitational waves decouple and can exist independently of each other.

  Relaxing the assumption that B is smooth, one can find (almost) purely gravitational impulsive wave configurations from the equation

$$\mu \sim \Delta^{(2)} B - B = \lambda \delta^{(2)}(\Omega - \Omega_0) \tag{6.36}$$

  where the δ-function represents a massless point source at fixed angle travelling along $\mathcal{N}$. This case has been considered in the context of black hole horizons and corrections to Hawking radiation in [11] (cf. also [25] for generalisations) and more recently in the context of the holography and scattering from black hole horizons e.g. in [12] and [13].

It thus appears that the more general expression for a shell with matter and impulsive gravitational wave that we have presented here (e.g. the simple configurations arising from BMS transformations) are smoothed out versions of this singular configuration and can perhaps be employed in the above contexts.

- Matter without Impulsive Gravitational Waves: $\hat{\gamma}_{ab} = 0$

  Finally we determine those soldering transformations that give rise to a shell that contains matter but no impulsive gravitational waves. We see immediately from (6.24) that a shell with no impulsive gravitational wave component must satisfy the hyperbolic equation

$$\nabla_\theta^{(2)} \partial_\theta F - \frac{1}{\sin^2 \theta} \nabla_\phi^{(2)} \partial_\phi F = 0 \tag{6.37}$$

  and the constraint

$$\nabla_\theta^{(2)} \partial_\phi F = F_{\theta\phi} - \frac{\cos\theta}{\sin\theta} F_\phi = 0 \; . \tag{6.38}$$

  The general solution to these two equations is

$$F(V, \theta, \phi) = A(V) + B(V)\; \vec{d}.\vec{r}(\theta, \phi) \tag{6.39}$$

  for an arbitrary constant vector $\vec{d}$, with $\vec{r}$ the unit vector on the 2-sphere,

$$\vec{r}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \; . \tag{6.40}$$

  Thus absence of an impulsive gravitational wave implies that the coordinate transformation $V_+ = F(V, \theta, \phi)$ only includes monopole and dipole terms, a plausible result (but we should note that this simple relation between dipole soldering transformations and absence of gravitational radiation holds only in Kruskal coordinates, not e.g. in Eddington-Finkelstein coordinates).

  In particular, choosing $T(\theta, \phi)$ in (6.29) to be an eigenfunction $Y_{\ell,m}$ of $\Delta^{(2)}$ with $\ell \geq 2$, say, one obtains a configuration with $p = j_A = 0$, but $\mu \neq 0$, accompanied by an impulsive gravitational wave travelling along the shell.

While we only presented this one special case of horizon shells in detail, it is of course straightforward to apply the procedure outlined above to other horizon shells, e.g. Reissner-Nordstrøm, or joining the Schwarzschild black hole to an AdS Schwarzschild black hole etc. In the cases we have looked at, including extremal Reissner-Nordstrøm, we have found no special new features beyond those already encountered in the above Schwarzschild - Kruskal example.

7 Horizon shell in Eddington-Finkelstein coordinates

7.1 General construction

We will now briefly also look at the Schwarzschild Horizon Shell in Eddington-Finkelstein coordinates,

$$ds^2 = -f_{ss}(r)dv^2 + 2dvdr + r^2 d\Omega^2 \;, \qquad f_{ss}(r) = 1 - \frac{2m}{r} \;. \tag{7.1}$$

In this case it is natural to make the choices $\Phi = r - 2m$ and $n = \partial_v$. Of course one can easily recalculate the γ_{ab} from scratch with these choices. It is obviously more efficient, however, to make use of the results of the previous section and to simply transform them from Kruskal to Eddington-Finkelstein coordinates. However, there is one subtlety that one needs to pay attention to.

Namely, since Kruskal V and the Eddington-Finkelstein advanced coordinate v are non-trivially related by

$$V = \mathrm{e}^{v/4m} \;, \qquad \frac{\partial v}{\partial V} = \frac{4m}{V} \;, \tag{7.2}$$

so are the corresponding normal vectors,

$$\partial_V = \frac{\partial v}{\partial V}\partial_v = \frac{4m}{V}\partial_v \;. \tag{7.3}$$

This leads to an opposite rescaling of the transverse vector N, thus of the γ_{ab}, and therefore to appropriate different rescaling of μ, j_A, and p which are homogeneous in n of degree 0,1,2 respectively (cf. also section 3.11.5 of [4]). Thus for example, the correct expression for the pressure in Eddington-Finkelstein coordinates is obtained not just by writing the Kruskal soldering transformation as

$$V_+ = F(V,\theta,\phi) = \mathrm{e}^{v_+/4m} = \mathrm{e}^{f(v,\theta,\phi)} \tag{7.4}$$

and substituting this into (6.19), but it requires an additional factor of $V/4m$. Proceeding either way one finds for the γ_{ab}

$$\begin{aligned}
\gamma_{vv} &= \frac{8m}{f_v}\left(\mathrm{e}^{-f/4m}\,\partial_v^2\,\mathrm{e}^{f/4m}\right) - \frac{1}{2m} \\
\gamma_{vA} &= \frac{8m}{f_v}\left(\mathrm{e}^{-f/4m}\,\partial_v\partial_A\,\mathrm{e}^{f/4m}\right) \\
\gamma_{AB} &= \frac{8m}{f_v}\left(\mathrm{e}^{-f/4m}\,\nabla^{(2)}_A\partial_B\,\mathrm{e}^{f/4m} + \frac{1}{2}\sigma_{AB}(f_v - 1)\right)
\end{aligned} \tag{7.5}$$

and then

$$\begin{aligned}
\mu &= -\frac{1}{8m\pi f_v}\left(\mathrm{e}^{-f/4m}\Delta^{(2)}\mathrm{e}^{f/4m} + f_v - 1\right) \\
j^A &= \frac{1}{64m^2\pi}\sigma^{AB}\left(2\frac{f_{vB}}{f_v} + \frac{f_B}{2m}\right) \\
p &= -\frac{1}{8\pi}\left(\frac{f_{vv}}{f_v} + \frac{f_v}{4m} - \frac{1}{4m}\right) \;.
\end{aligned} \tag{7.6}$$

The one non-trivial conservation law (5.17) is now

$$(\partial_v + \tilde{\kappa})\mu + \left(\nabla_A^{(2)} + \frac{1}{2}\gamma_{Av}\right) j^A + \frac{1}{4}\left(g^{AB}\gamma_{AB} - \frac{4}{m}\right) p = 0 \,, \tag{7.7}$$

which can be obtained directly from (5.17) or alternatively by transforming (6.20) according to the above prescription. This equation is again of course identically satisfied though the calculation to check this is somewhat lengthier than that in Kruskal coordinates. The angular components of the conservation law are just the obvious Eddington-Finkelstein counterpart of (6.22).

7.2 Special cases

- BMS Supertranslations of v

  One potentially interesting special class of soldering transformations are horizon BMS supertranslations of v [14–16]

  $$v \to f(v, \theta, \phi) = v + t(\theta, \phi) \tag{7.8}$$

  In this case the energy density, current and pressure are

  $$p = 0 \,, \qquad j^A = \frac{1}{128m^3\pi}\sigma^{AB}\partial_B t \,, \qquad \mu = -\frac{1}{8m\pi}\mathrm{e}^{-t/4m}\Delta^{(2)}\mathrm{e}^{t/4m} \tag{7.9}$$

  Even though the conservation law (7.7) is non-trivial in this case, because of the presence of both μ and the currents j^A, it is nevertheless true (and evident from the above expression for μ), that the energy density is conserved along the shell, $\partial_v\mu = 0$, as in the case of Kruskal horizon supertranslations discussed in section 6.2. Therefore we can again define an associated conserved energy $E[t]$ given (after an integration by parts) by

  $$E[t] = \int \sqrt{|g_{AB}|}d^2x \; \mu = -\frac{1}{8\pi}\int_{S^2}\frac{1}{4m}\sigma^{AB}\partial_A t \partial_B t \,. \tag{7.10}$$

  Since, when translated back to Kruskal coordinates, this transformation

  $$V_+ = V\mathrm{e}^{t/4m}, \tag{7.11}$$

  is not of the dipole form (6.39), this energy density and flux will almost invariably be accompanied by an impulsive gravitational shock wave.

- Comparison to calculations in [1]

  As our final illustration of the formalism, we look at a special case of the general formulation to make contact with some simple solderings considered in [1]. The general setting is that of two spherically symmetric space-times

  $$ds_\pm^2 = -\mathrm{e}^{2\psi_\pm} f_\pm dv^2 + 2\mathrm{e}^{\psi_\pm} dv dr + r^2 d\Omega^2. \tag{7.12}$$

  joined along their (common) static horizon at $r = r_s$, where $f_\pm(r_s) = 0$. Restricting to the case for which the interior and exterior geometries are those of the Schwarzschild black hole, $f_\pm(r) = f_{ss}(r)$, and choosing the exterior metric to be of the standard form (with $\psi_- = 0$), the non-trivial soldering is encoded in a non-trivial $\psi_+(v) = \psi(v)$, leading to a discontinuity of the metric in these coordinates. In [1] it is then shown how the energy-momentum tensor of the shell can be determined from the transverse extrinsic curvatures of the metrics on the 2 sides.

Alternatively, this interior form of the metric can be obtained by taking the interior metric in standard Eddington-Finkelstein coordinates ($\psi = 0$) and then applying the soldering transformation

$$v_+ = f(v) = \int^v \mathrm{e}^{\psi} dv \,. \tag{7.13}$$

From (7.6) one then deduces immediately that this shell carries non-zero energy density and pressure given by

$$\mu = \frac{1}{8\pi m}\left[\mathrm{e}^{-\psi}\right], \quad p = -\frac{1}{8\pi}\left[\partial_v\psi + \kappa_0 \mathrm{e}^{\psi}\right] \tag{7.14}$$

($\kappa_0 = 1/4m$ is the surface gravity of the Schwarzschild black hole), in complete agreement with equation (68) of [1].

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